Berry's phase for compact Lie groups.

E. Strahov*

Physics Dept., Technion-Israel Institute of Technology

Haifa, 32000 Israel

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Abstract

The Lie group adiabatic evolution determined by a Lie algebra parameter dependent Hamiltonian is considered. It is demonstrated that in the case when the parameter space of the Hamiltonian is a homogeneous Kähler manifold its fundamental Kähler potentials completely determine Berry geometrical phase factor. Explicit expressions for Berry vector potentials (Berry connections) and Berry curvatures are obtained using the complex parametrization of the Hamiltonian parameter space. A general approach is exemplified by the Lie algebra Hamiltonians corresponding to $SU(2)$ and $SU(3)$ evolution groups.

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*strahov@physics.technion.ac.il

I. Introduction

A number of branches of physics make use of the geometric properties of Kähler coset spaces (a definition may be found in Kobayashi and Nomizu, vol.2[1]). For instance, in quantum field theory the Kähler coset spaces give rise to a broad class of supersymmetric non-linear sigma models discussed in Zumino[2], Alvarez-Gaumé and Freedman[3], Bando, Kuramoto, Maskawa and Uehara[4] (among others). In quantization of dynamical systems with curved phase space with a non-trivial global geometry Kähler cosets serve as a model of such curved phase space (e.g. Beresin[5], Bar-Moshe and Marinov[6],[7]). Kähler geometry is also used in relativity theory (for a review see Flaherty[8]).

In this work the sphere of use of geometric properties of the Kähler coset spaces is extended to usual non-relativistic quantum mechanics. I show that knowledge of the fundamental Kähler potentials of these spaces enables to find a phase factor acquired by the quantum state under a compact group adiabatic evolution.

Berry[9] was the first to discover the relation between the adiabatic phase acquired by the wave function under a slow variation of the Hamiltonian parameters and the geometry of the parameter space. Specifically, it has been demonstrated that the adiabatic phase includes a part of a pure geometrical origin (the geometric phase factor). Simon[10] has shown that the geometrical meaning of the geometric phase is the holonomy in a Hermitian line bundle over the parameter space of the Hamiltonian, and that the adiabatic theorem[13] (see also Messiah[14]) gives rise to a connection with such bundle. When the parameter dependence of the Hamiltonian is determined by a closed curve C on the parameter space, the Berry geometrical factor Ω is expressed by the integral (Simon[10], Berry[9])

$$\Omega(C) = \int_S F. \tag{1}$$

Here S is any oriented surface in the parameter space with $\partial S = C$, and F is a two-form given on this parameter space. As a consequence of the Stock's theorem the two-form F may be expressed in terms of the Berry vector potential[9] (or Berry connection).

However, explicit forms for the geometric phase factor Ω (Eq.1) and for the Berry connection in terms of the (local) coordinates of the parameter space have since been obtained only for a number of simple cases. The spin precession in a slowly time-dependent magnetic field when the parameter space is a two-dimensional sphere, and the Berry connection is expressed in terms of the spherical coordinates is the simplest example. After a suitable reparametrization of the time variable the Hamiltonian H for this case may be chosen as

$$H(s) = \mathbf{n}(s)\mathbf{J}, \quad |\mathbf{n}(s)| = 1, \tag{2}$$

where $\mathbf{J}$ is the spin operator, and the vector $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. (It has been shown by Jakšić and Segert[11] that any two-level system may be described by the Hamiltonian (2) (with $\mathbf{J} = \boldsymbol{\sigma}$, $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices) after corresponding reparametrization of the time variable.) The corresponing Berry connection A_s induced by the adiabatic evolution of the spin state is:

$$A_s = \frac{l}{2}(1 - \cos\theta)\dot{\phi}, \quad l = 0, \pm1, \pm2 \ldots. \tag{3}$$

The Berry geometrical phase factor is defined by Eq.(1), where the two-form F is

$$F = \frac{l}{2}\sin\theta d\phi \wedge d\theta, \quad l = 0, \pm1, \pm2 \ldots. \tag{4}$$

In this example the evolution operator acting on the spin state belongs to an irreducible representation of the $SU(2)$ group, the Hamiltonian $H(s)$ (Eq.2) determines a smooth curve in the Lie algebra $su(2)$, and the parameter space is the homogeneous space of the group $SU(2)$, $S^2 = SU(2)/U(1)$.

It is a goal of the present paper to consider cases when the adiabatic evolution is determined by one-parameter Hamiltonians belonging to more complicated than $su(2)$ Lie algebras. I shall concentrate on one-parameter Hamiltonians which lead to compact group evolution operators, and determine closed smooth curves in the semi-simple Lie algebras of arbitrary ranks. The aim is to generalize equations Eq.(3), Eq.(4), and to find explicit expressions for Berry connections and Berry curvatures in terms of the coordinates of the corresponding parameter spaces.

This paper is organized as follows. Section II starts with an introduction of the relevant Lie algebra notations. We first obtain that the adiabatic phase factor is a scalar product of two vectors in the root space of the Lie algebra under consideration. We reveal that it is not always the case that the adiabatic phase factor depends on one integer only (as in the case of $su(2)$ Lie algebra, Eqs.(3), (4)). Rather, it is dependent on a set of integers with a number equal to the rank of the Lie algebra (e.g. in the case of $su(2)$ the rank is equal to one, thus only one integer suffices). We note that these integers determine irreducible representation in which quantum states form a basis.

In Section III a complex parametrization of the parameter space by a Mackey-type decomposition for any element of the evolution group G is introduced. Applicability of this procedure is restricted to the cases when the parametric space of the Hamiltonian is a homogeneous Kähler manifold G/H. Since in the cases under consideration this restriction is always satisfied, it becomes possible to apply this method to find explicit expressions for the adiabatic phase factor and the Berry potential in terms of the coordinates of the parameter space (Section IV). We discover that the Berry potentials may be expressed in terms of the fundamental Kähler potentials of the homogeneous Kähler manifold G/H. Bando, Kuramoto, Maskawa, and Uehara's[4] method is used to express the fundamental Kähler potentials in terms of the coordinates of the parameter space. Thus, the explicit expressions for Berry connections in terms of the complex parameters are found. This result will be formulated as a theorem in Section IV. It will be demonstrated in Section 5 that the action of the group G on the Kähler manifold G/H induces the gauge transformation of the Berry potentials. Once explicit forms for the Berry connections are obtained, the Berry curvature and the Berry geometrical phase factor are easily derived (Section V).

Specific cases of $SU(2)$ and $SU(3)$ evolution groups are considered in Section VI. Section VII concludes the paper.

II. Preliminaries and notations

Assume that a matrix irreducible representation of a compact semi-simple Lie group G of order n and rank r is given. Let $\mathcal{G}$ be the Lie algebra of G in which $\mathcal{H} \in \mathcal{G}$ denotes its Cartan subalgebra. A canonical Cartan-Weyl basis $\{h_j, e_\alpha, e_{-\alpha}\}$ in $\mathcal{G}$ is introduced , where $j = 1, \ldots, r \equiv \text{rank } \mathcal{G}$, and $\{\alpha\} \in \mathbf{\Delta}^+_{\mathcal{G}}$ are the positive roots of $\mathcal{G}$. (The definitions and properties of semi-simple Lie algebras and Lie groups may be found in Gilmore[12].) The number of the positive roots is $n_+ = \frac{1}{2}[n - r]$. The canonical basis of the Lie algebra $\mathcal{G}$ may be chosen so that the commutation relations will be written in the following standard form:

$$[h_i h_j] = 0 \quad , \quad [h_i, e_\alpha] = \alpha_i e_\alpha$$

$$[e_\alpha, e_{-\alpha}] = \sum_{j=1}^{r} \alpha_i h_i \quad , \quad [e_\alpha, e_\beta] = \chi(\alpha,\beta) e_{\alpha+\beta}. \tag{5}$$

Here $\chi(\alpha,\beta)$ is a function on the root lattice which vanishes if $\alpha+\beta \notin \Delta^+_{\mathcal{G}}$. Choosing primitive roots $\boldsymbol{\gamma}_j$, $j = 1,\ldots,r$, the fundamental weights $\boldsymbol{\omega}_j$, $j = 1,\ldots,r$ are determined from the equation

$$(\boldsymbol{\omega}_i \boldsymbol{\gamma}_i) = \frac{\delta_{ij}}{2}(\boldsymbol{\gamma}_i \boldsymbol{\gamma}_i). \tag{6}$$

For any unitary irreducible group representation its dominant weight $\mathbf{l}$ is given by a sum of the fundamental weights with nonnegative integer coefficients:

$$\mathbf{l} = \sum_{j=1}^{r} l_j \boldsymbol{\omega}_j = \sum_{j=1}^{r} \tilde{l}_j \mathbf{x}_j, \tag{7}$$

where $(\tilde{l}_1,\ldots,\tilde{l}_r)$ are the coordinates of the dominant weight vector $\mathbf{l}$ in the Lie algebra root space in which an orthogonal coordinate system is chosen. (Here and afterwards bold face is used to denote vectors in the Lie algebra root space.) The set $\{\mathbf{x}_j, j = 1,\ldots,r\}$ denotes the unit basis vector of this coordinate system.

We are interested in the cyclic adiabatic evolution of the dominant weight vector eigenket $\psi_{\mathbf{l}}$ which is defined by the following equation:

$$h_j \psi_{\mathbf{l}} = \tilde{l}_j \psi_{\mathbf{l}}, \quad j = 1,\ldots,r \tag{8}$$

(For convenience we are dealing with the dominant weight vector eigenket $\psi_{\mathbf{l}}$, corresponding to the dominant weight vector $\mathbf{l}$ here. The adiabatic evolution of an arbitrary weight vector eigenket may be considered in the same manner.) This adiabatic evolution will be determined by the Schrödinger equation with a Lie algebra Hamiltonian $b(s) \in \mathcal{G}$ given in the irreducible representation $(l_1,\ldots,l_r)$ of $\mathcal{G}$:

$$i\dot{\psi}(s) = \tau b(s)\psi(s), \quad \psi(0) = \psi_{\mathbf{l}}. \tag{9}$$

(The physical time t is replaced here by the scale time $s = t/\tau$, $s \in [0,1]$. The adiabatic limit is $\tau \to \infty$. The Hamiltonian $b(s)$ is assumed to depend smoothly on $s \in [0,1]$.) The cyclic evolution means that $b(s)$ takes the same values at the ends of the segment $[0,1]$. In order for the initial state $\psi_{\mathbf{l}}$ defined by Eq.(8) to be an eigenstate of the Hamiltonian $b(s)$, we demand that $b(0) = b(1) \in \mathcal{H}$.

The problem (9) can be written in terms of the Cartan-Maurer one form:

$$dgg^{-1} = -i\tau b(s)ds, \quad g(s) \in \mathsf{G}, \quad b(s) \in \mathcal{G}. \tag{10}$$

Here $g(s)$ is the Lie group evolution operator in the irreducible representation $(l_1, \ldots, l_r)$, $\psi(s) = g(s)\psi_1$ and $g(0) = e$ is the unit element of the compact evolution group G. Geometrically, the given Lie algebra Hamiltonian $b(s)$ determines a closed smooth curve in the Lie algebra $\mathcal{G}$ which begins and finishes in the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$. To solve Eq.(10) means to find the corresponding curve on the group manifold G.

For any given s the Hamiltonian $b(s) \in \mathcal{G}$ may be reduced to the Cartan subalgebra $\mathcal{H}$,

$$b(s) = g_1(s)\beta(s)g_1^{-1}(s), \quad \beta(s) \in \mathcal{H}. \tag{11}$$

It is usefull to assume that the Cartan subalgebra element β does not depend on the parameter $s \in [0, 1]$, i.e. the eigenvalues of the Hamiltonian $b(s)$ are constants on the segment under considerations. For example, the $su(2)$ Lie algebra Hamiltonian given by Eq.(2) has two constant eigenvalues ± 1 if $\mathbf{J} = \boldsymbol{\sigma}$, $(\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. When $b(s)$ determines the closed curve that begins and finishes in $\mathcal{H}$, $g_1(0) = g_1(1) = e$. Assuming that the conditions of the adiabatic theorem are satisfied, the probabilities of transitions induced by a parameter dependence of $b(s)$ are supressed and the initial and final quantum states are distinct in the phase factor only, the unknown group element $g(s)$ is decomposed as

$$g(s) = g_1(s)h(s), \quad h(s) \in \mathsf{H}. \tag{12}$$

($\mathsf{H} \in \mathsf{G}$ denotes the Cartan subgroup of the Lie group G). Inserting the decomposed elements $b(s)$ (Eq.11) and $g(s)$ (Eq.12) into Eq.(10) we obtain:

$$dhh^{-1} = -i\tau\beta - g_1^{-1}dg_1. \tag{13}$$

The unknown Cartan subgroup element $h(s) \in \mathsf{H}$ depends on r real parameters $\Theta^j(s)$,

$$h(s) = \exp\left(-i\sum_{j=1}^{r}\Theta^j(s)h_j\right). \tag{14}$$

The parameters $\Theta^j(s)$ are obtained from Eq.(13) using the orthogonality condition for the Cartan subalgebra basis elements $\mathrm{Tr}(h_i h_j) = \delta_{ij}$. After a cyclic evolution ($s = 1$) we find

$$\Theta^j(1) = \tau\beta^j - i\int_0^1 \mathrm{Tr}\left(g_1^{-1}\frac{dg_1}{ds}h_j\right)ds. \tag{15}$$

The second term in the above expression is real and will determine the geometric part of the adiabatic phase acquired by the dominant weight vector eigenket $\psi_{\mathbf{l}}$ after the cyclic adiabatic evolution.

Let us introduce the r-dimensional vectors in the root space of the Lie algebra $\mathcal{G}$:

$$\boldsymbol{\beta} = \sum_{j=1}^{r} \beta^j \mathbf{x}_j, \quad \boldsymbol{\mathcal{Q}} = \sum_{j=1}^{r} \mathcal{Q}^j \mathbf{x}_j. \tag{16}$$

The components $\mathcal{Q}^j$ of the vector $\boldsymbol{\mathcal{Q}}$ are defined by the second term in Eq.(15), i.e.

$$\mathcal{Q}^j = -i \int_0^1 \mathrm{Tr}\left(g_1^{-1} \frac{dg_1}{ds} h_j \right) ds. \tag{17}$$

After a cyclic adiabatic evolution the dominant weight vector eigenket $\psi_{\mathbf{l}}$ is multiplied by the element of the Cartan subgroup $h(1)$ given by Eqs.(14),(15), i.e.

$$\psi_{\mathbf{l}} \rightarrow h(1)\psi_{\mathbf{l}} = \exp\left(-i \sum_{j=1}^{r} \Theta^j(1) h_j \right) \psi_{\mathbf{l}}. \tag{18}$$

Using Eq.(8) we finally obtain that the adiabatic phase factor Θ acquired by the quantum state $\psi_{\mathbf{l}}$ is given by the scalar product of the dominant weight vector $\mathbf{l}$ (Eq.7) on the sum of the vectors $\boldsymbol{\mathcal{Q}}$ and $\boldsymbol{\beta}$:

$$\psi_{\mathbf{l}} \rightarrow \exp\left(-i\Theta\right) \psi_{\mathbf{l}}, \quad \Theta = \mathbf{l} \cdot \boldsymbol{\beta} + \mathbf{l} \cdot \boldsymbol{\mathcal{Q}}. \tag{19}$$

While the first term in the above expression for the adiabatic phase factor Θ is associated with the dynamical phase, the second term $\mathbf{l} \cdot \boldsymbol{\mathcal{Q}}$ is the geometrical phase factor Ω, defined as

$$\Omega \equiv \mathbf{l} \cdot \boldsymbol{\mathcal{Q}} \tag{20}$$

As we can see from Eq.(19), the Berry geometrical phase factor depends on integers $l_1, \ldots, l_r$ which determine the dominant weight vector $\mathbf{l}$ (Eq.7), and characterize the irreducible representation of the evolution group under consideration.

III. Complex parametrization of the Hamiltonian parameter space.

The given Lie algebra Hamiltonian $b(s) \in \mathcal{G}$ varies adiabatically through a circuit C in the parameter space which is the homogeneous group manifold G/H. Indeed, $b(s)$ depends on s

only through the group element $g_1(s)$ (Eq.11). The Hamiltonian $b(s)$ decomposition Eq.(11) is invariant under $g_1(s) \to g_1(s)h_1, \quad \forall h_1 \in \mathsf{H}$, so $g_1(s)$ must be chosen as a representative of the corresponding equivalence class. Thus, there is a gauge freedom in the diagonalization process (Eq.11), and the Cartan subgroup $\mathsf{H} \in \mathsf{G}$ is the group of the gauge transformations. Geometrically, G is described as a principal fiber bundle with the Cartan subgroup H as the standard fiber and G/H is the base coset space. According to Borel's theorem[15], the necessary and sufficient condition for the coset space G/H (where G is a compact semi-simple group, and H is a closed subgroup of G) to be a homogeneous Kähler manifold is that H be the centrelizer of a torus in G. A torus means a direct product of any $U(1)$ subgroup of H and the centralizer means a subgroup which consists of all G elements commutative with that torus elements. As it may be seen from Section 2, in the case under consideration conditions of the Borel theorem are satisfied (H is a Cartan subgroup commuting with a torus), so the Hamiltonian parameter is a homogeneous Kähler manifold. Then the complex parametrization on G/H may be introduced, and the explicit expression for the geometrical factor Ω may be found in terms of the (complex) coordinates of the parameter space G/H.

The desired complex parametrization on the homogeneous group manifold G/H is introduced by the complex parametrization of the group element g_1, which determines decomposition of the Hamiltonian $b(s)$ (Eq.11). Namely, given the canonical basis, the Lie algebra $\mathcal{G}$ is split into three subalgebras, $\mathcal{G} = \mathcal{H} \oplus \mathcal{B}_+ \oplus \mathcal{B}_-$, ($\mathcal{B}_+, \mathcal{B}_-$ are called Borel subalgebras), corresponding to three subsets of the basis elements $\{e_{-\alpha}\}, \{h_j\}, \{e_\alpha\}$. Respectively, the Lie algebra $\mathcal{B}_+(\mathcal{B}_-)$ generates a nilpotent Borel subgroup $\mathsf{B}_+(\mathsf{B}_-) \subset \mathsf{G}^c$ (G^c means the complexification of the group G). The element g_1 has a unique (left) Mackey decomposition

$$g_1 = ug_-, \quad u \in \mathsf{B}_+, \quad g_- \in \mathsf{G}/\mathsf{B}_+. \tag{21}$$

(Note that in order to get u for any given g_1 one has to impose the condition that $g_- = u^{-1}g_1$ has no part in B_+. That would determine u completely). The complex parameters which can be introduced in G/H correspond to the positive roots of $\mathcal{G}$,

$$u(z) = \exp\left(\sum_{\alpha\in\Delta^+_{\mathcal{G}}} z^\alpha e_\alpha\right), \quad z^\alpha \in \mathcal{C}. \tag{22}$$

$u(z)$ is an element of a nilpotent group and its matrix representions are polynominals of z^α.

The local form (22) for $u(z)$ is valid in a neighborhood of the point $z^\alpha = 0$, i.e. the origin of the coordinate system in G/H. The origin is related to the choice of coordinates. A transition to other domains of G/H covering the Kähler manifold G/H may be performed by the group transformation.

Given $u(z)$, $g_-(z,\bar z)$ will acquire the following form:

$$g_-(z,\bar z) = v^+(z,\bar z)k(z,\bar z), \quad v(z,\bar z) \in \mathsf{B}_+, \quad k(z,\bar z) \in \mathsf{H}. \tag{23}$$

The elements $v^+(z,\bar z), k(z,\bar z)$ are expressed as exponentials of the corresponding Lie algebra elements:

$$v^+(z,\bar z) = \exp\Big(\sum_{\alpha\in\Delta^+_{\mathcal{G}}} y^\alpha(z,\bar z)e_{-\alpha}\Big), \quad k(z,\bar z) = \exp\Big(\sum_{i=1}^{r} \kappa^i(z,\bar z)h_i\Big). \tag{24}$$

For a particular $u(z) \in \mathsf{B}_+$ the functions $y^\alpha(z,\bar z)$ and $\kappa^i(z,\bar z)$ may be determined when the group element g_1 is unitary:

$$g_1^+ = g_1^{-1} \to v^+kk^+v = (u^+u)^{-1}. \tag{25}$$

v is obtained from the (right) Mackey decomposition of $(u^+u)^{-1}$, and the explicit forms for the functions $y_\alpha(z,\bar z)$ may be found. As soon as v is given, one turns to calculation of k from the equation:

$$kk^+ = (vu^+uv^+)^{-1} \in \mathsf{H} \tag{26}$$

The functions $\kappa^i(z,\bar z)$ are especially important as we shall see below. It will be shown in Section 4 that the functions $\kappa^i(z,\bar z)$ completely determine the Berry potentials when G is a compact evolution group, and the Hamiltonian parameter space is G/H. These functions are lineary related with the fundamental Kähler potentials $K^i(z,\bar z)$ of the Kähler manifold G/H under considerations:

$$K^i(z,\bar z) = -2\sum_{j=1}^{r} \kappa^j(z,\bar z)\mathrm{Tr}(h_j\eta_i). \tag{27}$$

The formula (27) was obtained by Itoh, Kugo and Kunimoto[16]. Here η_i are the projection matrices introduced by Bando, Kuratomo, Maskawa and Uehara[4]. The projection matrices exist in any matrix representation of G and correspond to elements of the Cartan subalgebra $h_j \in \mathcal{H}$. The basic properties of the projection matrices are [4]:

$$\begin{aligned} &\eta_j = \eta_j^+ \quad \eta_j^2 = \eta_j \quad \eta_j\hat h_k = \hat h_k\eta_j \quad \forall j,k = 1,\dots,r \\ &\eta_j\hat e_{-\alpha}\eta_j = \hat e_{-\alpha}\eta_j \qquad \eta_j\hat e_\alpha\eta_j = \eta_j\hat e_\alpha \end{aligned} \tag{28}$$

(The hat stands for the matrix representation.) All η_j are commuting with each other. For any representation of G, where $\hat{h}_j$ are diagonal, all η_j are also diagonal. The explicit forms of η_j satisfying Eq.(28) may be found (Bando, Kuratomo, Maskawa and Uehara[4]). For the irreducible representation under consideration the functions $\kappa^j(z,\bar{z}),\ j=1,\ldots,r$ may be expressed lineary in terms of the fundamental Kähler potentials $K^j(z,\bar{z}),\ \ j=1,\ldots,r$ (Eq.27). In its turn a suitable method of construction of the fundamental Kähler potentials is given by Bando, Kuratomo, Maskawa and Uehara[4](see also Itoh, Kugo and Kunitomo[16]). A number of particular examples is considered by Marinov and Bar-Moshe[6], [7] in relation to the geometric quantization on homogeneous compact Kähler manifolds.

A technique for constructing the fundamental Kähler potentials may be described as follows. Once the projection matrices η_j are obtained from Eqs.(28), the projected determinant is defined for any matrix M as

$$\det{}_{\eta_j} M \equiv \det(\eta_j M \eta_j + I - \eta_j). \tag{29}$$

For any projection matrix η_j, a fundamental Kähler potential $K^j(z,\bar{z})$ is constructed from the fundamental represenation for the element $u(z)$ (Eq.22) of the nilpotent Borel subgroup B_+,

$$K^j(z,\bar{z}) \equiv \ln \det{}_{\eta_j}\Big(u(z)^+ u(z)\Big). \tag{30}$$

Note that the fundamental Kähler potential $K^j(z,\bar{z})$ is not a global function on $\mathsf{G/H}$, except for cases where $\mathsf{G/H}$ has a trivial topology. However, the manifold $\mathsf{G/H}$ may be covered with complex coordinate neighborhoods. A transition from one neighborhood to another may be given by the group transformation. If the group G acts holomorphically on $\mathsf{G/H}$, $z \to gz, \forall g \in \mathsf{G}$, the fundamental Kähler potentials (29) are transformed as

$$K^j(gz,\overline{gz}) = K^j(z,\bar{z}) + \Phi^j(z,g) + \overline{\Phi^j(z,g)}, \tag{31}$$

where $\Phi^j(z,g)$ are locally holomorphic functions of $z^\alpha,\ \ \alpha=1,\ldots,\frac{n-r}{2}$. These functions must satisfy the following cocycle condition,

$$\Phi^j(z,g_2g_1) = \Phi^j(g_2z,g_1) + \Phi^j(z,g_2),\ \ \forall g_1,g_2 \in \mathsf{G} \tag{32}$$

which results from the group property $z \to g_2(g_1z) = (g_2g_1)z$.

IV. Expression of Berry connection in terms of the fundamental Kähler potentials.

With all the preliminary steps completed, we are in a position to formulate the main result of this work.

Theorem. Suppose that the cyclic adiabatic evolution of the dominant weight vector eigenket $\psi_{\mathbf{l}}$ defined by Eq.(8) is determined by the Shrödinger equation (Eq.9). Let the Hamiltonian parameter space be a compact homogeneous Kähler manifold G/H, where G is the compact evolution group, and H its Cartan subgroup. Then the geometrical phase factor Ω acquired by the quantum state $\psi_{\mathbf{l}}$ is

$$\Omega = \int_0^1 A_s ds, \tag{33}$$

where the Berry connection A_s is completely determined in terms of the fundamental Kähler potentials of the parameter space G/H. Explicitly, when the local complex parametrization $\left\{z^\alpha, \overline{z^\alpha}, \;\; \alpha = 1, \ldots, \frac{n-r}{2}\right\}$ on G/H is introduced,

$$A_s = \mathbf{l} \cdot \mathbf{A}(z, \bar{z}), \quad \mathbf{A}(z, \bar{z}) = \mathcal{L}_{z,\bar{z}} \boldsymbol{\kappa}(z, \bar{z}). \tag{34}$$

$\mathbf{A}(z, \bar{z})$ and $\boldsymbol{\kappa}(z, \bar{z})$ are the vectors in the root space of the Lie algebra of G given in the orthonormal basis $\{\mathbf{x}_j, \;\; j = 1, \ldots, r\}$

$$\boldsymbol{\kappa}(z, \bar{z}) = \sum_{j=1}^{r} \kappa^j(z, \bar{z}) \mathbf{x}_j, \quad \mathbf{A}(z, \bar{z}) = \sum_{j=1}^{r} A^j(z, \bar{z}) \mathbf{x}_j. \tag{35}$$

$\mathcal{L}_{z,\bar{z}}$ is the (hermitian) differential operator:

$$\mathcal{L}_{z,\bar{z}} = i \sum_{\alpha, \bar{\alpha}=1}^{\frac{n-r}{2}} \left(\dot{z}^\alpha \partial_\alpha - \dot{z}^{\bar{\alpha}} \partial_{\bar{\alpha}} \right), \tag{36}$$

where $\dot{z} \equiv \frac{dz(s)}{ds}$, $\;z^{\bar{\alpha}} \equiv \overline{z^\alpha}$, $\;\partial_\alpha \equiv \frac{\partial}{dz^\alpha}$, $\;\partial_{\bar{\alpha}} \equiv \frac{\partial}{dz^{\bar{\alpha}}}$. The real functions $\kappa^j(z, \bar{z})$ define the Cartan subgroup element $k(z, \bar{z}) \in \mathsf{H}$ (Eq.24) under (left) Mackey decomposition of a representative of the coset space G/H (Eqs.21-24). These functions are lineary connected with the fundamental Kähler potentials $K^j(z, \bar{z})$ (Eq.27).

Proof: The geometric phase factor Ω is given by the scalar product of the dominant weight vector $\mathbf{l}$ and the vector $\boldsymbol{\mathcal{Q}}$ (Eq.19). The coordinates of the vector $\boldsymbol{\mathcal{Q}}$ in the root space of the Lie algebra $\mathcal{G}$ corresponding to the evolution group G are determined by Eq.17. When the local complex parametrization on the coset space G/H is introduced, the components $\mathcal{Q}^j$ of the vector $\boldsymbol{\mathcal{Q}}$ may be represented as a sum of two integrals:

$$\mathcal{Q}^j = -i \sum_{\alpha,\bar{\alpha}=1}^{\frac{n-r}{2}} \left\{ \int_C \mathrm{Tr}\left(g_1^{-1}\partial_\alpha g_1 h_j\right) dz^\alpha + \int_C \mathrm{Tr}\left(g_i^{-1}\partial_{\bar{\alpha}} g_1 h_j\right) dz^{\bar{\alpha}} \right\}. \tag{37}$$

The group element $g_1(z,\bar{z})$ which is the representative of the coset space G/H is decomposed (see Section 3) as

$$g_1(z,\bar{z}) = u(z) v^+(z,\bar{z}) k(z,\bar{z}), \tag{38}$$

where $u(z) \in \mathsf{B}_+,\;\; v^+(z,\bar{z}) \in \mathbf{B}_-,\;\; k(z,\bar{z}) \in \mathsf{H}$ are given by Eqs.(22), (24). Let us recall that

$$k(z,\bar{z}) = \exp\left(\sum_{j=1}^{r} \kappa^j(z,\bar{z}) h_j\right).$$

It may be shown (see, for example, Itoh, Kugo and Kunimoto[16]) that

$$\mathrm{Tr}\left(g_1^{-1}\partial_\alpha g_1 h_j\right) = -\partial_\alpha \kappa^j(z,\bar{z}), \quad \mathrm{Tr}\left(g_1^{-1}\partial_{\bar{\alpha}} g_1 h_j\right) = \partial_{\bar{\alpha}} \kappa^j(z,\bar{z}). \tag{39}$$

Indeed, noting that

$$\partial_{\bar{\alpha}} g_1(z,\bar{z}) = u(z)\partial_{\bar{\alpha}}\Big(v^+(z,\bar{z})k(z,\bar{z})\Big), \tag{40}$$

we find that

$$\mathrm{Tr}\left(g_1^{-1}\partial_\alpha g_1 h_j\right) = \mathrm{Tr}\Big(h_j (v^+(z,\bar{z}))^{-1} \partial_{\bar{\alpha}}(v^+(z,\bar{z}))\Big) + \sum_{i=1}^{r} \partial_\alpha(\kappa^i(z,\bar{z}))\mathrm{Tr}(h_i h_j). \tag{41}$$

The expression $\Big(v^+(z,\bar{z})\Big)^{-1}\partial_{\bar{\alpha}}\Big(v^+(z,\bar{z})\Big)$ produces only terms belonging to the Borel subalgebra $\mathcal{B}_-$. As a consequence, the first term in the above equation equals to zero. Using the orthogonality condition for the Cartan subalgebra canonic basis elements

$$\mathrm{Tr}(h_i h_j) = \delta_{ij}, \tag{42}$$

we obtain

$$\mathrm{Tr}\left(g_1^{-1}\partial_{\bar{\alpha}} g_1 h_j\right) = \partial_{\bar{\alpha}} \kappa^j(z,\bar{z}). \tag{43}$$

In order to prove the first equation in (39) we use

$$\begin{aligned} g_1^{-1}\partial_\alpha g_1 &= g_1^+\partial_\alpha\left(g_1^+\right)^{-1} = k^+vu^+\partial_\alpha\Big((u^+)^{-1}v^{-1}(k^+)^{-1}\Big) \\ &= k^+v\partial_\alpha\Big(v^{-1}(k^+)^{-1}\Big), \end{aligned} \tag{44}$$

Afterwards, we proceed with the proof as in the previous case.

From Eqs.(37), (39) we find

$$\mathcal{Q}^j = i\sum_{\alpha,\bar\alpha=1}^{\frac{n-r}{2}}\int_0^1 \left(\dot z^\alpha\partial_\alpha - \dot z^{\bar\alpha}\partial_{\bar\alpha}\right)\kappa^j(z,\bar z)ds \equiv \int_0^1 A^j(z,\bar z)ds. \tag{45}$$

Thus, the vector in the root space of $\mathcal{G}$

$$\mathbf{A}(z,\bar z) = i\sum_{\alpha,\bar\alpha=1}^{\frac{n-r}{2}} \left(\dot z^\alpha\partial_\alpha - \dot z^{\bar\alpha}\partial_{\bar\alpha}\right)\boldsymbol{\kappa}(z,\bar z) \equiv \mathcal{L}_{z,\bar z}\boldsymbol{\kappa}(z,\bar z) \tag{46}$$

is introduced, the Berry connection and the Berry geometrical phase factor are determined by Eqs.(33), (34) respectively. □

In the next section we use this result to demonstrate that the holomorphic action of the evolution group G on the Hamiltonian parameter space $\mathsf{G/H}$ induces the gauge transformation of the Berry potentials. In addition, the Berry curvature and the Berry geometrical phase factor will be obtained.

V. Gauge transformation and Berry curvature

Consider the transformation of the vector $\mathbf{A}(z,\bar z)$ under the holomorphic action of the group G on the homogeneous Kähler $\mathsf{G/H}$. As soon as the fundamental Kähler potentials $K^j(z,\bar z)$, $j = 1,\ldots,r$ are transformed in accordence with Eq.(31), the vector $\boldsymbol{\kappa}(z,\bar z)$ changes in a similar fashion, i.e.

$$\boldsymbol{\kappa}(z,\bar z) \to \boldsymbol{\kappa}(gz,\overline{gz}) = \boldsymbol{\kappa}(z,\bar z) + \boldsymbol{\phi}(g,z) + \overline{\boldsymbol{\phi}(g,z)}. \tag{47}$$

Indeed, given the decomposition of the coset space representative (Eqs.21-24), the action of an arbitrary group element $g_2 \in \mathsf{G}$ on the coset space $\mathsf{G/H}$ is defined (by Coleman, Wess and Zumino[17]) as

$$g_2u(z) = u(g_2z)g_-(z,g_2), \tag{48}$$

and $g_2 z$ is a rational function of z. Once the nonlinear realization of the group action on the coset space $\mathsf{G/H}$ is determined (Eq.48), the the transformation law (47) may be proved using the Mackey-type decomposition of the product $g_1 \cdot g_2$ (where $g_1 \in \mathsf{G/H}$ is given by Eqs.(21)-(24))(for further details see Itoh, Kogo and Kunimoto[16]. The change of the real vector $\boldsymbol{\kappa}(z,\bar{z})$ (Eq.47) under the holomorphic action of the group on its coset space leads to the gauge transformation of the vector $\mathbf{A}(z,\bar{z})$:

$$\mathbf{A}(z,\bar{z}) = \mathcal{L}_{z,\bar{z}}\boldsymbol{\kappa}(z,\bar{z}) \rightarrow \mathbf{A}(gz,\overline{gz}) = \mathbf{A}(z,\bar{z}) + d\mathbf{W}(z,\bar{z}), \tag{49}$$

where the real vector $\mathbf{W}(z,\bar{z})$ is defined in terms of the vectors $\boldsymbol{\phi}(g,z)$, $\overline{\boldsymbol{\phi}(g,z)}$:

$$\mathbf{W}(z,\bar{z}) \equiv i\Big(\boldsymbol{\phi}(g,z) - \overline{\boldsymbol{\phi}(g,z)}\Big). \tag{50}$$

Respectively, the abelian Berry connection A_s defined by Eq.(34) is transformed as

$$\begin{aligned} A_s(z,\bar{z}) \rightarrow A_s(gz,\overline{gz}) &= A_s(z,\bar{z}) + dW(z,\bar{z}) \\ W(z,\bar{z}) &\equiv \mathbf{l}\cdot\mathbf{W}(z,\bar{z}). \end{aligned} \tag{51}$$

Note that the expression (33) for Ω may be rewritten as

$$\Omega = \sum_{\alpha=1}^{\frac{n-r}{2}} \int_C A_\alpha(z,\bar{z})dz^\alpha + \sum_{\alpha=1}^{\frac{n-r}{2}} \int_C A_{\bar{\alpha}}(z,\bar{z})dz^{\bar{\alpha}} \tag{52}$$

where

$$\begin{aligned} A_\alpha(z,\bar{z}) &\equiv i\partial_\alpha\Big(\mathbf{l}\cdot\boldsymbol{\kappa}(z,\bar{z})\Big), \\ A_{\bar{\alpha}}(z,\bar{z}) &\equiv -i\partial_{\bar{\alpha}}\Big(\mathbf{l}\cdot\boldsymbol{\kappa}(z,\bar{z})\Big). \end{aligned} \tag{53}$$

Under the holomorphic action of the group G on the homogeneous Kähler manifold $\mathsf{G/H}$, $A_\alpha(z,\bar{z})$, $A_{\bar{\alpha}}(z,\bar{z})$ transform as

$$\begin{aligned} A_\alpha(z,\bar{z}) &\rightarrow A_\alpha(gz,\overline{gz}) = A_\alpha(z,\bar{z}) + i\partial_\alpha\Big(\mathbf{l}\cdot\boldsymbol{\phi}(g,z)\Big) \\ A_{\bar{\alpha}}(z,\bar{z}) &\rightarrow A_{\bar{\alpha}}(gz,\overline{gz}) = A_{\bar{\alpha}}(z,\bar{z}) - i\partial_{\bar{\alpha}}\Big(\mathbf{l}\cdot\bar{\boldsymbol{\phi}}(g,z)\Big). \end{aligned} \tag{54}$$

Using the Stokes theorem, we obtain the expression for the Berry geometrical factor in terms of the surface integral,

$$\Omega = \int_S F, \quad F = \sum_{\alpha,\bar{\beta}=1}^{\frac{n-r}{2}} \frac{\partial^2 K^{(\mathbf{l})}(z,\bar{z})}{\partial z^\alpha \partial z^{\bar{\beta}}} dz^\alpha \wedge dz^{\bar{\beta}}, \tag{55}$$

where

$$K^{(\mathbf{l})}(z,\bar{z}) = 2\Big(\mathbf{l}\cdot\boldsymbol{\kappa}(z,\bar{z})\Big), \tag{56}$$

and S is any oriented surface in the parameter space G/H with $\partial S = C$. As it may be seen from Eqs.(47), (55), the Berry curvature F is invariant under the gauge transformation (Eq.51) induced by the holomorphic group action on the parameter space G/H.

A simple way to calculate the vector $\boldsymbol{\kappa}(z,\bar{z})$ which determines the Berry connection and the Berry curvature is to use Eq.(27). This formula connects the vector $\boldsymbol{\kappa}(z,\bar{z})$ with the fundamental Kähler potentials given by Eqs.(28)-(30).

VI. Examples of adiabatic evolutions induced by Lie groups

$SU(2)$ adiabatic evolution

The space $S^2 = SU(2)/U(1)$ is the simplest homogeneous Kähler manifold. The generators of the group $SU(2)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{57}$$

The elements of the canonical basis of the Lie algebra $SU(2)$ in the fundamental (spinor) representation are

$$E_1 = \frac{1}{2}(\sigma_1 + i\sigma_2); \quad E_2 = \frac{1}{2}(\sigma_1 - i\sigma_2); \quad H = \frac{1}{\sqrt{2}}\sigma_3. \tag{58}$$

Correspondingly, the canonical commutation relations for the $SU(2)$ Lie algebra have the following form:

$$[e_1, e_2] = \sqrt{2}h, \quad [h, e_1] = \sqrt{2}e_1, \quad [h, e_2] = -\sqrt{2}e_2. \tag{59}$$

The rank of the Lie algebra $SU(2)$ is equal to one, $r = 1$. The root space is one-dimensional, R^1, with two opposite roots $\pm\boldsymbol{\alpha} = \pm\sqrt{2}\mathbf{x}$, and the primitive root is $\boldsymbol{\gamma} = \boldsymbol{\alpha}$. The fundamental weight vector $\boldsymbol{\omega}$ is determined by Eq.(6), $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\alpha}$. An arbitrary irreducible representation is defined by the dominant weight vector,

$$\mathbf{l} = l\boldsymbol{\omega} = \frac{l}{\sqrt{2}}\mathbf{x}, \quad l = 0, \pm 1, \pm 2, \ldots. \tag{60}$$

We consider the cyclic adiabatic evolution of the dominant weight vector eigenket $\psi_{\mathbf{l}}$ which is the eigenvector of the following eigenvalue problem:

$$h\psi_{\mathbf{l}} = \frac{l}{\sqrt{2}}\psi_{\mathbf{l}}, \quad l = 0, \pm 1, \pm 2, \dots . \tag{61}$$

In accordance with Eqs.(19), (15) after the cyclic adiabatic evolution the state $\psi_{\mathbf{l}}$ acquires the geometrical phase factor Ω (Eqs.17, 20). The abelian Berry connection A_s is given by (34). The vector $\boldsymbol{\kappa}$ in the case of $SU(2)$ evolution group may be obtained by the (left) Mackey-type decomposition of a representative $g_1(z, \bar{z})$ of the coset space $SU(2)/U(1)$:

$$g_1(z, \bar{z}) = u(z) g_-(z, \bar{z}), \quad u(z) = \exp(z e_1). \tag{62}$$

The element $g_-(z, \bar{z})$ is found in accordance with the general procedure described in Section III:

$$\begin{aligned} g_-(z, \bar{z}) &= \exp\Big((y(z, \bar{z}) e_2\Big) \exp\Big(\kappa(z, \bar{z}) h\Big) \\ y(z, \bar{z}) &= -\frac{\bar{z}}{1 + z\bar{z}}, \quad \kappa(z, \bar{z}) = \frac{1}{\sqrt{2}} \ln(1 + z\bar{z}). \end{aligned} \tag{63}$$

Therefore the vector (in the root space of the Lie algebra $su(2)$) $\boldsymbol{\kappa}$ is given by

$$\boldsymbol{\kappa}(z, \bar{z}) = \frac{1}{\sqrt{2}} \ln(1 + z\bar{z}) \mathbf{x}, \tag{64}$$

and the abelian Berry connection A_s is expressed in terms of the complex coordinates of the coset space $SU(2)/U(1)$ as:

$$A_s = \frac{il}{2} \left(\frac{\dot{z}\bar{z} - \dot{\bar{z}} z}{1 + z\bar{z}} \right). \tag{65}$$

Using the Stokes theorem we determine the Berry geometrical phase factor (Eqs.17, 20):

$$\Omega = -il \iint \frac{dz \wedge d\bar{z}}{(1 + z\bar{z})^2}, \quad l = 0, \pm 1, \pm 2, \dots . \tag{66}$$

In order to compare Eq.(66) with the original Berry result[9] for the spin procession in the time-dependent magnetic field, use the stereographic projection of the two-dimensional sphere with the unit radius :

$$|z| = \cot \theta/2 \quad \arg z = \varphi. \tag{67}$$

Then the geometrical phase factor is equal to

$$\omega = \frac{l}{2} \iint \sin \theta d\theta \wedge d\varphi, \quad l = 0, \pm 1, \pm 2, \dots . \tag{68}$$

It is this result that was obtained by Berry[9].

$SU(3)$ adiabatic evolution

The Cartan subgroup of the $SU(3)$ group is $U(1)\times U(1)$, so the Berry geometrical phase factor will be determined by the geometry of the Flag manifold $SU(3)/U(1)\times U(1)$. The canonical basis of $SU(3)$ Lie algebra is introduced with the help of the eight Gell-Mann generators:

$$\lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\qquad \lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\qquad \lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$
$$\lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}\qquad \lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\qquad \lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\tag{69}$$
$$\lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad \lambda_8=\tfrac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

Then the elements of the canonical Cartan-Weyl basis in the fundamental three-dimensional representation are given by

$$\begin{array}{rclrclrcl}E_{12}&=&1/2(\lambda_1+i\lambda_2); & E_{23}&=&1/2(\lambda_6+i\lambda_7); & E_{13}&=&1/2(\lambda_4+i\lambda_5);\\ E_{21}&=&1/2(\lambda_1-i\lambda_2); & E_{32}&=&1/2(\lambda_6-i\lambda_7); & E_{31}&=&1/2(\lambda_4-i\lambda_5);\\ H_1&=&\frac{\sqrt{3}}{2}\lambda_3+\frac{\lambda_8}{2}; & H_2&=&-\frac{\lambda_3}{2}+\frac{\sqrt{3}}{2}\lambda_8. & & & \end{array}\tag{70}$$

The rank of the Lie algebra $SU(3)$ is equal to two, $r=2$. The root space is two dimensional, R^2, and the canonical commutation relations determining the positive root vectors are

$$\begin{array}{rclcrcll}[h_1,e_{12}]&=&\frac{3}{\sqrt{6}}e_{12}&;&[h_2,e_{12}]&=&-\frac{1}{\sqrt{2}}e_{12}&;\\ [h_1,e_{13}]&=&\frac{3}{\sqrt{6}}e_{13}&;&[h_2,e_{13}]&=&\frac{1}{\sqrt{2}}e_{13}&;\\ [h_1,e_{23}]&=&0&;&[h_2,e_{23}]&=&-\sqrt{2}e_{23}&.\end{array}\tag{71}$$

From the commutation relations (71) we find six non-zero root vectors,

$$\pm\boldsymbol{\alpha_1}=\pm\left(\frac{3}{\sqrt{6}};-\frac{1}{\sqrt{2}}\right),\quad \pm\boldsymbol{\alpha_2}=\pm\left(\frac{3}{\sqrt{6}};\frac{1}{\sqrt{2}}\right),\quad \pm\boldsymbol{\alpha_3}=\left(0;-\sqrt{2}\right).\tag{72}$$

The root diagram is a hexagon, the two primitive roots are $\boldsymbol{\gamma_1}=\boldsymbol{\alpha_3}$, $\boldsymbol{\gamma_2}=\boldsymbol{\alpha_2}$. The fundamental weight are found to be

$$\boldsymbol{\omega_1}=\left(\frac{1}{\sqrt{6}};-\frac{1}{\sqrt{2}}\right),\quad \boldsymbol{\omega_2}=\left(\frac{2}{\sqrt{6}};0\right).\tag{73}$$

An arbitrary irreducible representation is defined by the (two-dimensional) dominant weight vector:

$$\mathbf{l} = l_1\boldsymbol{\omega_1} + l_2\boldsymbol{\omega_2}, \quad l_1, l_2 = 0, \pm1, \pm2, \ldots. \tag{74}$$

The coordinates of this vector in the root space are:

$$\tilde{l}_1 = \frac{l_1}{\sqrt{6}} + \frac{2l_2}{\sqrt{6}}, \quad \tilde{l}_2 = -\frac{l_1}{\sqrt{2}}. \tag{75}$$

The dominant weight vector eigenket $\psi_{\mathbf{l}}$ is an eigenvector of both Cartan Lie algebra elements h_1, h_2 with eigenvalues $\tilde{l}_1, \tilde{l}_2$. It follows from the Theorem (Section 4) that in order to find the geometric phase factor Ω acquired by the dominant weight vector eigenket $\psi_{\mathbf{l}}$, one should determine the two-component vector $\mathbf{k}(z,\bar{z})$ in the root space of the Lie algebra $su(3)$. As it may be seen from Eq.(27) the components of this vector in the orthogonal basis of the root space are linear combinations of the fundamental Kähler potentials $K^1(z,\bar{z}),\ K^2(z,\bar{z})$ of the homogeneous Kähler manifold $SU(3)/U(1)\times U(1)$. The fundamental Kähler potentials $K^1(z,\bar{z}),\ K^2(z,\bar{z})$ are constructed using Eq.(30), where the element $u(z)$ is taken in the fundamental 3×3 representation:

$$u(z) \equiv \exp\Big(z_1E_{12} + z_2E_{23} + z_3E_{13}\Big) = \begin{pmatrix} 1 & z_1 & z_3^+ \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{76}$$

We have denoted $z^\pm = z_3 \pm \frac{1}{2}z_1z_2$, and the projection matrices found from Eqs.(28) are

$$\eta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{77}$$

Knowing $u(z), \eta_1, \eta_2$ in the fundamental representation, the fundamental Kähler potentials are calculated, and we obtain

$$K^1(z,\bar{z}) = \ln(1 + z_1\bar{z}_1 + z_3^-\bar{z}_3^-), \quad K^2(z,\bar{z}) = \ln(1 + z_2\bar{z}_2 + z_3^+\bar{z}_3^+). \tag{78}$$

Finding the coefficients $\mathrm{Tr}(h_j\eta_i)$ (in the fundamental represenation $h_1 \equiv H_1,\ h_2 \equiv H_2$), we get the components of the vector $\boldsymbol{\kappa}(z,\bar{z})$ in the root space of $su(3)$ algebra:

$$\kappa^1(z,\bar{z}) = \frac{\sqrt{6}}{4}K^1(z,\bar{z}), \quad \kappa^2(z,\bar{z}) = \frac{1}{\sqrt{2}}K^2(z,\bar{z}) - \frac{1}{2\sqrt{2}}K^1(z,\bar{z}). \tag{79}$$

(Note that the expressions for the components ofthe vector $\boldsymbol{\kappa}(z,\bar{z})$ in terms of the complex coordinates Eqs.(78), (79) may be obtained also by the (left) Mackey decomposition of the coset space representative g_1, Eqs.(21)-(24)).

For the case of $SU(3)$ evolution group the Berry connection A_s given by Eq.(34) is (80):

$$\begin{aligned} A_s &= \mathcal{L}_{z,\bar{z}}\left\{\mathbf{l}\cdot\boldsymbol{\kappa}(z,\bar{z})\right\}, \quad \mathbf{l}\cdot\boldsymbol{\kappa}(z,\bar{z}) = \frac{1}{2}\Big[(l_1+l_2)K^1(z,\bar{z}) - l_1K^2(z,\bar{z})\Big] \\ \mathcal{L} &= \sum_{\alpha,\bar{\alpha}=1}^{3}\left(\dot{z}^\alpha\frac{\partial}{\partial z^\alpha} - \dot{z}^{\bar{\alpha}}\frac{\partial}{\partial z^{\bar{\alpha}}}\right), \end{aligned} \tag{80}$$

and the Berry curvature F (Eq.55) is

$$F = \sum_{\alpha,\bar{\beta}=1}^{3}\frac{\partial^2K^{(\mathbf{l})}(z,\bar{z})}{\partial z^\alpha\partial z^{\bar{\beta}}}dz^\alpha\wedge dz^{\bar{\beta}}, \tag{81}$$

where the real function $K^{(\mathbf{l})}(z,\bar{z})$ is a linear combination of the fundamental Kähler potentials $K_1(z,\bar{z}),\ \ K_2(z,\bar{z})$ with the integer coefficients:

$$K^{(\mathbf{l})}(z,\bar{z}) = (l_1+l_2)K^1(z,\bar{z}) - l_1K^2(z,\bar{z}), \quad l_1,l_2 = 0,\pm1,\pm2\ldots. \tag{82}$$

Note that a common approach to the $SU(3)$ group evolution is to use the Euler coordinates that are similar to the Euler angle parameters of $SU(2)$. Such method has been used by Byrd[18], Arvind, Mallesh and Mukunda[19], Khanna, Mukhopadhaya, Simon, and Mukunda[20] in connection with the evolution of a three level system. While the geometric phase factor for this case has been found, none of them noticed that in cases under their consideration there exists an intimate relation between the geometric phase factors, Berry connections and Berry curvature and the fundamental Kähler potentials of parameter spaces.

VII. Conclusions.

In this paper we have considered the adiabatic evolution determined by a compact Lie group evolution operator taken in an arbitrary irreducible representation. It has been demonstrated that when the parameter space of the Hamiltonian is a homogeneous Kähler manifold its fundamental Kähler potentials completely determine Berry geometrical phase factor. Besides, we have shown that Berry geometrical factor and the Berry connections depend on a set of integers a number of which equals to the rank of the corresponding Lie algebra. These integers determine irreducible representation in which quantum states form a basis.

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